

Sums of Squares Bijective Parameter Representation

Walter Wyss

Abstract

A pair of n -dimensional vectors $x, y \in \mathbb{R}^n$ of equal length

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n), & y &= (y_1, y_2, \dots, y_n) \\ x_1^2 + x_2^2 + \dots + x_n^2 &= y_1^2 + y_2^2 + \dots + y_n^2 \end{aligned} \quad (1)$$

generate a convenient parameterization that uniquely reproduces the vectors. If the components of the vectors are rational numbers, the parameters are rational and vice versa. Multiplying with the greatest common denominator we get the complete solution, up to scaling and absolute values, of equation (1) in positive integers.

We give examples where some of the components are equal or coincide. We also apply our representation to the parallelogram equation

$$2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \quad (2)$$

and to the equation

$$x^2 + y^2 + z^2 = 3w^2 \quad (3)$$

1 Parameterization.

For $x, y \in \mathbb{R}^n$

$$x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n)$$

we have the scalar product

$$(x, y) = \sum_{i=1}^n x_i y_i \quad (4)$$

and the norm $\|x\|$ through

$$\|x\|^2 = \sum_{i=1}^n x_i^2 \quad (5)$$

Equation (1) means that the vectors x and y have equal length

$$\|x\| = \|y\| \quad (6)$$

Equivalently to (1) we have

$$x_1^2 - y_1^2 + \cdots + x_k^2 - y_k^2 + \cdots + x_n^2 - y_n^2 = 0 \quad (7)$$

We now introduce the quantities s_i and d_i by

$$x_i + y_i = 2s_i, \quad x_i - y_i = 2d_i, \quad i = 1, 2, \cdots, n \quad (8)$$

giving

$$x_i = s_i + d_i, \quad y_i = s_i - d_i \quad (9)$$

In vector notation

$$s = (s_1, \cdots, s_n), \quad d = (d_1, \cdots, d_n) \quad (10)$$

we have

$$x = s + d, \quad y = s - d \quad (11)$$

From (7) we find

$$(s, d) = 0 \quad (12)$$

$$\|x\|^2 = \|s\|^2 + \|d\|^2 \quad (13)$$

Thus the vectors s and d are orthogonal.

Given the vector

$$s = (s_1, \dots, s_n), \quad s_1 \neq 0 \quad (14)$$

we look at the vectors, $k = 2, 3, \dots, n$

$$\begin{aligned} e_2 &= (-s_2, s_1, 0, \dots, 0) \\ e_k &= (-s_k, 0, \dots, s_1, 0, \dots, 0), \quad s_1 \text{ in the } k^{\text{th}} \text{ place} \\ e_n &= (-s_n, 0, \dots, 0, s_1) \end{aligned} \quad (15)$$

These are (n-1) vectors that are linearly independent and orthogonal to the vector s . Thus they span the vector space orthogonal to the vector s . Since the vector d is orthogonal to the vector s , it can be written as

$$d = \sum_{k=2}^n \lambda_k e_k \quad (16)$$

The parameterization of (1) is now given by

$$x = s + \sum_{k=2}^n \lambda_k e_k \quad (17)$$

$$y = s - \sum_{k=2}^n \lambda_k e_k \quad (18)$$

or in components

$$\begin{aligned} x_1 &= s_1 - \sum_{k=2}^n \lambda_k s_k \\ x_k &= s_k + \lambda_k s_1, \quad k = 2, \dots, n \end{aligned} \quad (19)$$

$$\begin{aligned} y_1 &= s_1 + \sum_{k=2}^n \lambda_k s_k \\ y_k &= s_k - \lambda_k s_1, \quad k = 2, \dots, n \end{aligned} \quad (20)$$

The (2n-1) parameters $s_1, \dots, s_n, \lambda_2, \dots, \lambda_n$ are now our convenient parameters giving us the components $x_i, y_i, i = 1, \dots, n$.

Conversely, having the components x_i, y_i , we find the parameters by

$$2s_i = x_i + y_i, \quad i = 1, \dots, n \quad (21)$$

$$2s_1\lambda_k = x_k - y_k, \quad k = 2, \dots, n \quad (22)$$

If the components x_i, y_i are rational, the parameters s_i, λ_k are rational and if the parameters s_i, λ_k are rational the components x_i, y_i are rational due to (19) and (20). There is a one-to-one correspondence between the parameters and the components.

2 Examples.

a)
$$x_1^2 + x_2^2 = y_1^2 + y_2^2 \quad (23)$$

$$\begin{aligned} x_1 &= s_1 - \lambda_2 s_2 & y_1 &= s_1 + \lambda_2 s_2 \\ x_2 &= s_2 + \lambda_2 s_1 & y_2 &= s_2 - \lambda_2 s_1 \end{aligned} \quad (24)$$

Conversely

$$2s_1 = x_1 + y_1, \quad 2s_2 = x_2 + y_2 \quad (25)$$

$$2s_1\lambda_2 = x_2 - y_2 \quad (26)$$

b)
$$x_1^2 + x_2^2 = y_1^2 \quad (27)$$

Then $y_2 = 0$, resulting in $s_2 = \lambda_2 s_1$ and thus

$$x_1 = s_1[1 - \lambda_2^2], \quad y_1 = s_1[1 + \lambda_2^2] \quad (28)$$

$$x_2 = 2s_1\lambda_2 \quad (29)$$

Conversely

$$2s_1 = x_1 + y_1, \quad 2s_1\lambda_2 = x_2 \quad (30)$$

$$c) \quad x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 \quad (31)$$

Then $y_k = 0, k = 2, \dots, n$, resulting in $s_k = \lambda_k s_1$
and thus

$$\begin{aligned} x_1 &= s_1[1 - \lambda_2^2 - \cdots - \lambda_n^2], & y_1 &= s_1[1 + \lambda_2^2 + \cdots + \lambda_n^2] \\ x_k &= 2s_1\lambda_k, & k &= 2, \dots, n \end{aligned} \quad (32)$$

Conversely

$$2s_1 = x_1 + y_1, \quad 2s_1\lambda_k = x_k, \quad k = 2, \dots, n \quad (33)$$

3 Applications.

a) Parallelogram equation [1]

$$2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \quad (34)$$

We could use the parameterization (19) (20) for $n=4$, but it is more convenient to introduce

$$2u_+ = u_4 + u_3, \quad 2u_- = u_4 - u_3 \quad (35)$$

Then

$$u_3 = u_+ - u_-, \quad u_4 = u_+ + u_- \quad (36)$$

and

$$u_1^2 + u_2^2 = u_+^2 + u_-^2 \quad (37)$$

We now use (23) and get the bijective parameter representation

$$u_1 = s_1 - \lambda_2 s_2, \quad u_+ = s_1 + \lambda_2 s_2 \quad (38)$$

$$u_2 = s_2 + \lambda_2 s_1, \quad u_- = s_2 - \lambda_2 s_1 \quad (39)$$

or

$$u_3 = s_1 + \lambda_2 s_2 - s_2 + \lambda_2 s_1 \quad (40)$$

$$u_4 = s_1 + \lambda_2 s_2 + s_2 - \lambda_2 s_1 \quad (41)$$

Conversely, from (24)

$$2s_1 = u_1 + u_+, \quad 2s_2 = u_2 + u_-, \quad 2s_1\lambda_2 = u_2 - u_- \quad (42)$$

or

$$4s_1 = 2u_1 + u_4 + u_3, \quad 4s_2 = 2u_2 + u_4 - u_3 \quad (43)$$

$$4s_1\lambda_2 = 2u_2 - u_4 + u_3 \quad (44)$$

Now introducing m, n, u through

$$s_1 = u, \quad s_2 = nu, \quad \lambda_2 = m \quad (45)$$

we find the bijective parameter representation of the parallelogram equation (34) in terms of the parameters m, n, u, as

$$\begin{aligned} u_1 &= (1 - mn)u \\ u_2 &= (m + n)u \\ u_3 &= (1 + mn - n + m)u \\ u_4 &= (1 + mn + n - m)u \end{aligned} \quad (46)$$

Conversely

$$\begin{aligned} 4u &= 2u_1 + u_4 + u_3 \\ m &= \frac{2u_2 - u_4 + u_3}{4u} \\ n &= \frac{2u_2 + u_4 - u_3}{4u} \end{aligned} \quad (47)$$

b) The equation

$$x^2 + y^2 + z^2 = 3w^2 \quad (48)$$

Our parameter representation (19) (20) gives

$$\begin{aligned} x &= s_1 - \lambda_2 s_2 - \lambda_3 s_3 & w &= s_1 + \lambda_2 s_2 + \lambda_3 s_3 \\ y &= s_2 + \lambda_2 s_1 & w &= s_2 - \lambda_2 s_1 \\ z &= s_3 + \lambda_3 s_1 & w &= s_3 - \lambda_3 s_1 \end{aligned} \quad (49)$$

Conversely

$$\begin{aligned} 2s_1 &= x + w, & 2s_2 &= y + w, & 2s_3 &= z + w \\ 2s_1\lambda_2 &= y - w, & 2s_1\lambda_3 &= z - w \end{aligned} \quad (50)$$

Equation (49) results in

$$\lambda_3 = \lambda_2 + \frac{s_3 - s_2}{s_1} \quad (51)$$

and with

$$q = s_1 + s_2 + s_3 \quad (52)$$

in

$$\lambda_2 = \frac{1}{q} \left[s_2 - s_1 + s_3 \frac{s_2 - s_3}{s_1} \right] \quad (53)$$

Then

$$\begin{aligned} w &= s_2 - \lambda_2 s_1 \\ w &= \frac{1}{q} [s_1^2 + s_2^2 + s_3^2] \end{aligned} \quad (54)$$

and with

$$p = s_1 s_2 + s_2 s_3 + s_3 s_1 \quad (55)$$

we find with (50) and the parameters s_1, s_2, s_3

$$\begin{aligned}
x &= 2s_1 - w \\
x &= \frac{1}{q}[2p + q(s_1 - s_2 - s_3)]
\end{aligned} \tag{56}$$

$$\begin{aligned}
y &= 2s_2 - w \\
y &= \frac{1}{q}[2p + q(s_2 - s_3 - s_1)]
\end{aligned} \tag{57}$$

$$\begin{aligned}
z &= 2s_3 - w \\
z &= \frac{1}{q}[2p + q(s_3 - s_1 - s_2)]
\end{aligned} \tag{58}$$

Up to scaling and with (52) (55) we find all solutions of (48) by

$$\begin{aligned}
p &= s_1 s_2 + s_2 s_3 + s_3 s_1 \\
q &= s_1 + s_2 + s_3 \\
x &= 2p + q(s_1 - s_2 - s_3) \\
y &= 2p + q(s_2 - s_3 - s_1) \\
z &= 2p + q(s_3 - s_1 - s_2) \\
w &= s_1^2 + s_2^2 + s_3^2
\end{aligned} \tag{59}$$

Finally, introducing the six parameters

$m_1, n_1, m_2, n_2, m_3, n_3$, all integers, through

$$s_1 = \frac{m_1}{n_1}, \quad s_2 = \frac{m_2}{n_2}, \quad s_3 = \frac{m_3}{n_3},$$

we get all integer solutions of (48), up to scaling, with

$$\begin{aligned}
P &= m_1 n_1 m_2 n_2 n_3^2 + m_2 n_2 m_3 n_3 n_1^2 + m_3 n_3 m_1 n_1 n_2^2 \\
Q &= m_1 n_2 n_3 + m_2 n_3 n_1 + m_3 n_1 n_2
\end{aligned} \tag{60}$$

as follows

$$\begin{aligned}
x &= 2P + Q[m_1n_2n_3 - m_2n_3n_1 - m_3n_1n_2] \\
y &= 2P + Q[m_2n_3n_1 - m_3n_1n_2 - m_1n_2n_3] \\
z &= 2P + Q[m_3n_1n_2 - m_1n_2n_3 - m_2n_3n_1] \\
w &= [m_1n_2n_3]^2 + [m_2n_3n_1]^2 + [m_3n_1n_2]^2
\end{aligned} \tag{61}$$

Thus x, y, z, w are polynomials of degree six in six integer variables.

References

- [1] Walter Wyss, *Perfect Parallelograms*, American Math Monthly, 119 (6) (2012), p.513-515

Department of Physics, University of Colorado Boulder, Boulder, CO 80309
Walter.Wyss@Colorado.EDU